**$\psi$-Pascal and $\hat{q}_\psi$-Pascal matrices - an accessible factory of one source identities and resulting applications**

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**Summary**

Recently the author proposed two extensions of Pascal and $q$-Pascal matrices defined here also - in the spirit of the Ward "Calculus of sequences" [1] promoted in the framework of the $\psi$- Finite Operator Calculus [2,3] . Specifications to $q$-calculus case and Fibonomial calculus case are made explicit as an example of abundance of new possibilities being opened. In broader context the $\psi$-Pascal $P_\psi[x]$ and $\hat{q}_\psi$-Pascal $P_{\hat{q}_\psi}[x]$ matrices appear to be as natural as standard Pascal matrix $P[x]$ already is known to be [4]. Among others these are a one source factory of streams of identities and indicated resulting applications.

# 1 I. On the usage of references

The papers of main reference are: [1-3]. We shall take here notation from [2,3] (see below) and the results from [1] as well as from [2,3] - for granted. For other respective references see: [2,3].The acquaintance with "The matrices of Pascal and other greats" [4] is desirable. Further relevant references of the present author are: [5] on extended finite operator calculus of Rota and quantum groups and other [6-7]. The reference to q-Pascal matrix is [8] Further Pascal matrix references for further readings are [9-14]. One may track down there among others relations: the Pascal Matrix versus Classical Polynomials. The book [15] is recommended and the recent reference [16] is useful for further applications. Very recent $\psi$-Pascal matrix reference is [17] and also recent further Pascal matrices references ( far more not complete list of them ) are to be found in [18-21] . The book of Kassel Christian [22] - makes an intriguing link to the advanced world of related mathematics.

Before to proceed we anyhow explain -for the reader convenience - some of the very basic of the intuitively useful $\psi$-notation promoted by the author [2,3,5,6]. Here $\psi$ denotes an extension of $\langle \frac{1}{n!} \rangle_{n \geq 0}$ sequence to quite arbitrary one (the so called - admissible) and the specific choices are for example: Fibonomialy-extended $\langle \frac{1}{F_n!} \rangle_{n \geq 0}$ (here $\langle F_n \rangle$ denotes the Fibonacci sequence ) or Gauss $q$-extended $\langle \frac{1}{n_q!} \rangle_{n \geq 0}$ admissible sequences of extended umbral operator calculus or just "the usual" $\langle \frac{1}{n!} \rangle_{n \geq 0}$ common choice. We get used to write these $q$ - Gauss and other extensions in mnemonic convenient upside down notation [2,3,5,6]

$$\psi_n \equiv n_\psi, x_\psi \equiv \psi(x) \equiv \psi_x, n_\psi! = n_\psi (n-1)_\psi!, n > 0, \tag{1}$$

$$x_{\psi}^{\underline{k}} = x_\psi (x-1)_\psi (x-2)_\psi ... (x-k+1)_\psi \tag{2}$$

$$x_\psi (x-1)_\psi ... (x-k+1)_\psi = \psi(x)\psi(x-1)...\psi(x-k-1). \tag{3}$$

The corresponding $\psi$-binomial symbol and $\partial_\psi$ difference linear operator on F[[x]] (F - any field of zero characteristics) are below defined accordingly where following Roman [3,3,5,6] we shall call $\psi = \{\psi_n(q)\}_{n \geq 0}$; $\psi_n(q) \neq 0$; $n \geq 0$ and $\psi_0(q) = 1$ an *admissible sequence.*

**Definition 1** *The $\psi$-binomial symbol is defined as follows:*

$$\binom{n}{k}_{\psi} = \frac{n_{\psi}!}{k_{\psi}!(n-k)_{\psi}!} = \binom{n}{n-k}_{\psi}$$

**Definition 2** *Let $\psi$ be admissible. Let $\partial_{\psi}$ be the linear operator lowering degree of polynomials by one defined according to $\partial_{\psi} x^n = n_{\psi} x^{n-1}$; $n \geq 0$. Then $\partial_{\psi}$ is called the $\psi$-derivative.*

You may consult [2,3,5,6] and references therein for further development and use of this notation "$q$-commuting variables" - included.

## 2 II. Towards $\psi$-Pascal matrix factory of identities

Let us define analogously to [4,9,10] define the $\psi$-Pascal matrix as

$$P_{\psi}[x] = exp_{\psi}\{xK_{\psi}\}$$

where ($Z_n$ denotes the additive cyclic group)

$$K_{\psi} = \Big( (j+1)_{\psi} \delta_{i,j+1} \Big)_{i,j \in Z_n}$$

therefore

$$P_{\psi}[x] = \left( x^{i-j} \binom{i}{j}_{\psi} \right)_{i,j \in Z_n}$$

due to: $\partial_{\psi} P_{\psi}[x] = K_{\psi} \psi P_{\psi}[x]$ where $\psi P_{\psi}[x]|_{x=0} = K_{\psi}$.

Explicitly (see [8] for $q$-case) $K_{\psi}$ matrix is of the form

$$K_{\psi} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 7_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10_{\psi} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11_{\psi} & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (n-1)_{\psi} & 0
\end{bmatrix}$$

**Fig.1. The $K_{\psi}$ matrix**

Naturally $K_{\psi}^{n} = 0; K_{\psi}^{k} \neq 0$ for $0 \leq k \leq (n-1)$. Hence we have

$$P_{\psi}[x] = exp_{\psi}\{xK_{\psi}\} = \sum_{k \in Z_n} \frac{x^k K_{\psi}^k}{k_{\psi}!}$$

the result $P_\psi[x]$ of $\psi$- exponentiation above being shown on the Fig.2.

$$P_\psi[x] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^2 & 2_\psi x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^3 & 3_\psi x^2 & 3_\psi x & 1 & 0 & 0 & 0 & 0 & 0 \\ x^4 & 4_\psi x^3 & 6_\psi x^2 & 4_\psi x & 1 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & 0 \\ x^{n-1} & 0 & 0 & 0 & 0 & 0 & 0 & (n-1)_\psi x & 1 \end{bmatrix}$$

**Fig. 2. The $P_\psi[x]$ matrix**

Immediately we see that the $\psi$-Pascal matrix $P_\psi[x] = exp_\psi\{xK_\psi\}$ is also the source of many important identities. Here below there are the examples correspondent to those from [4] which are accordingly infered from the $\psi$-additivity property (non-group property in general) :

$$P_\psi[x]P_\psi[y] = P_\psi[x +_\psi y].$$

**Warning**: for not normal sequences : see: [1,2,3,5,6,8] - the one parameter family $\{P_\psi[x]\}_{x\in F}$ is *not a group* ! since for not normal sequences $(1 -_\psi 1)^{2k} \neq 0$ thought $[x +_\psi (-x)]^{2k+1} = 0$.

In general we are dealing with abelian semigroup with identity which becomes the group only for normal sequences. And so coming back to identities we have for example :

$$(4) \quad \sum_{j\leq k\leq i} \binom{i}{k}_\psi \binom{k}{j}_\psi = (1 +_\psi 1)^{i-j} \binom{i}{j}_\psi, i \geq j \Longleftrightarrow P_\psi[1]P_\psi[1] = P_\psi[1 +_\psi 1].$$

$$(5) \sum_{j\leq k\leq i} (-1)^k \binom{i}{k}_\psi \binom{k}{j}_\psi = (1 -_\psi 1)^{i-j} \binom{i}{j}_\psi, i \geq j \Longleftrightarrow P_\psi[1]P_\psi[-1] = P_\psi[1 -_\psi 1].$$

The above identities after the choice $\psi = \langle \frac{1}{n!} \rangle_{n\geq 0}$ coincide with the corresponding ones from [4]. There are much more examples of this nature.

We shall now try also to find out a kind of $\psi$-extended version of the $q$-identity (6)

$$(6) \qquad \sum_{0\leq k\leq i} \binom{i}{k}_q^2 = \binom{2i}{i}_q \Longleftrightarrow P_\psi[1]P_\psi^T[1] \equiv F_q[1].$$

where we have defined the $q$-Fermat matrix as follows

$$(7) \qquad F_q[1] = \left( \binom{i+j}{i}_q \right)_{i,j\in Z_n}.$$

For q =1 case- name Fermat - see [15] for this Fermat called Pascal symmetric Matrix for q =1 see: [4,9]. For $q$-binomial - see below in **Important**.

In order to find out a kind of $\psi$-extended version of the Pascal-Fermat $q$-identity identity (6) we shall proceed as in [16]. There the Cauchy $\hat{q}_\psi$- identity and $\hat{q}_\psi$-Fermat

matrix were introduced due to the use of the $\hat{q}_\psi$-muting variables from Extended Finite Operator Calculus [3,5]. The *linear* $\hat{q}_\psi$-mutator operator was defined in [3,5,16] as follows for $F$ - field of characteristic zero and $F[x]$ - the linear space of polynomials.

$$\hat{q}_\psi : F[x] \to F[x]; \quad \hat{q}_\psi x^n = \frac{(n+1)_\psi - 1}{n_\psi} x^n; \quad n \geq 0.$$

**Important**. With the Gaussian choice of admissible sequence [3,5]

$$\psi = \{\psi_n(q)\}_{n\geq 0}, \psi_n(q) = [n_q!]^{-1}, n_q = \frac{1-q^n}{1-q}, n_q! = n_q(n-1)_q!, 1_q! = 0_q! = 1, \hat{q}_\psi x^n = q^n x^n$$

and the $\hat{q}_\psi$-Pascal and $\hat{q}_\psi$-Fermat matrices from [16] (see next section ) coincide with $q$-Pascal and $q$-Fermat matrices correspondingly **which is not the case** for the general case - for example Fibonomial $F$-Pascal matrix is different from $\hat{q}_F$-Pascal matrix - see next section.

In [16] in analogy to the standard case [9,10,4] the matrices with operator valued matrix elements

$$x^{i-j}\binom{i}{j}_{\hat{q}_\psi}, \binom{i+j}{j}_{\hat{q}_\psi}$$

were named the $\hat{q}_\psi$- Pascal P[x] and $\hat{q}_\psi$ -Fermat F[1] matrices - correspondingly i.e.

$$P_{\hat{q}_\psi}[x] = \left( x^{i-j}\binom{i}{j}_{\hat{q}_\psi} \right)_{i,j\in Z_n}$$

The $\hat{q}_\psi$-P[1] Pascal and $\hat{q}_\psi$-F[1] Fermat matrices from [16] are related via the following identity for operator valued matrix elements

$$(8) \qquad \sum_{k\geq 0} \hat{q}_\psi^{(r-k)(j-k)} \binom{i}{k}_{\hat{q}_\psi} \binom{j}{k}_{\hat{q}_\psi} = \binom{i+j}{j}_{\hat{q}_\psi}.$$

The relation (8) is the one being looked for to extend the Pascal-Fermat $q$-identity (6). Here - following [16]- we use the new $\hat{q}_\psi$-Gaussian symbol with operator valued matrix elements.

**Definition 3** *We define $\hat{q}_\psi$-binomial symbol i.e. $\hat{q}_\psi$-Gaussian coefficients as follows:* $\binom{n}{k}_{\hat{q}_\psi} = \frac{n_{\hat{q}_\psi}!}{k_{\hat{q}_\psi}!(n-k)_{\hat{q}_\psi}!} = \binom{n}{n-k}_{\hat{q}_\psi}$ *where* $n_{\hat{q}_\psi}! = n_{\hat{q}_\psi}(n-1)_{\hat{q}_\psi}!, 1_{\hat{q}_\psi}! = 0_{\hat{q}_\psi}! = 1$ *and* $n_{\hat{q}_\psi} = \frac{1-\hat{q}_\psi^n}{1-\hat{q}_\psi}$ *for* $n > 0$.

# 3 III. Specifications : $q$-umbral and umbral Fibonomial cases

**III-q** $q$-umbral calculus case [1,2,3,5-8]

Let us make the $q$-Gaussian choice [2,3,5,6,8] of the admissible sequence $\psi = \{\psi_n(q)\}_{n\geq 0}$. Then the $\psi$-Pascal matrix becomes the $q$-Pascal matrix from [8] and we arrive mnemonic at the corresponding to $q = 1$ case numerous $q$-identities and

other "$q$-applications". Specifically in the $q$-case we have (see: Proposition 4.2.3 in [22])

$$\sum_{k\geq 0} q^{(r-k)(j-k)} \binom{r}{k}_q \binom{s}{j-k}_q = \binom{r+s}{j}_q \tag{9}$$

hence from this Cauchy $q$-identity we obtain the following easy to find out formula for the symmetric Pascal (or Fermat) matrix elements:

$$\sum_{k\geq 0} q^{(r-k)(j-k)} \binom{i}{k}_q \binom{j}{k}_q = \binom{i+j}{j}_q . \tag{10}$$

Naturally we are dealing now with not normal sequences i.e. not with a one parameter $q$-Pascal group [8] since for $(1 -_q 1)^{2k} \neq 0$ though $[x +_q (-x)]^{2k+1} = 0$ ; see: [1] and then [2,3,5,6,8]. If $q$-Pascal matrix $P_q[1] = exp_q\{xK_q\}|_{x=1}$ is considered also for $q \in GF(q)$ field then $q = p_m$ where $p$ is prime and $\binom{n}{k}_q$ becomes the number of $k$-dimensional subspaces in $n-th$ dimensional space over Galois field $GF(q)$. Also $q$ real and $-1 < q < +1$ cases are exploited in vast literature on the so-called $q$-umbral calculus (for Cigler , Roman and Others see: [3,23] and references therein- links to thousands in [23]). It is not difficult to notice that the $\hat{q}_\psi$- Pascal and $\hat{q}_\psi$-Fermat matrices under the $q$-Gassian choice of the admissible sequence $\psi$ - coincide with $q$-Pascal and $q$-Fermat matrices correspondingly which is meaningful **magnificent exception** and which is not the case in general .

**III-F FFOC**-umbral calculus case [6-7]

In straightforward analogy to the $q$-case above consider now the Fibonomial coefficients (see: FFOC = Fibonomial Finite Operator Calculus Example 2.1 in [6]) where $F_n$ denote the Fibonacci numbers and $\psi_n(q) = [F_n!]^{-1}$ .

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_F^{\underline{k}}}{k_F!}, \quad n_F \equiv F_n \neq 0,$$

where we make an analogy driven [6,5,3,2] identifications ($n > 0$):

$$n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F;$$

$$0_F! = 1; \quad n_F^{\underline{k}} = n_F(n-1)_F \ldots (n-k+1)_F.$$

**Information** In [7] a partial ordered set was defined in such a way that the Fibonomial coefficients count the number of specific finite "birth-self-similar" sub-posets of this infinite non-tree poset naturally related to the Fibonacci tree of rabbits growth process.

The $\psi$-Pascal matrix becomes then the $F$-Pascal matrix and we arrive at the corresponding $F$-identities (mnemonic replacement of $\psi$ by $F$ ) and other "$F$-applications" - hoped to be explored soon.

Naturally we are now dealing with **not normal** sequences : see: [1,2,3,5,6,8] -i.e.we have no $F$-Pascal **group** since for $(1 -_F 1)^{2k} \neq 0$ though $[x +_F (-x)]^{2k+1} = 0$. For example: $(x +_F y)^2 = x^2 + F_2xy + y^2, (x +_F y)^4 = x^4 + F_4x^3y + F_4F_3x^2y^2 + F_4xy^3 + y^4$.

Here in the Fibonomial choice case the semi-group generating matrix matrix $K_F$ is of the form

$$
K_F = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & F_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & F_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & F_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & F_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & F_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{(n-1)} & 0
\end{bmatrix}
$$

**Fig.3. The $K_F$ matrix**

and the corresponding beautiful $F$-Pascal matrix $P_F[x] = exp_F\{xK_F\}$ being $F$-exponentiation of $K_F$ reads:

$$
P_F[x] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^2 & F_2x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^3 & F_3x^2 & F_3x & 1 & 0 & 0 & 0 & 0 & 0 \\
x^4 & F_4x^3 & F_6x^2 & F_4x & 1 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & 0 \\
x^{n-1} & F_{n-1}x^{n-1} & \binom{n-1}{2}_F x^{n-2} & . & . & . & . & F_{(n-1)}x & 1
\end{bmatrix}
$$

**Fig. 4. The $P_F[x]$ matrix**

The $\hat{q}_F$-Pascal $P_{K_{\hat{q}_F}}[x]$ and $K_{\hat{q}_F}$-Fermat matrix do not coincide with $F$-Pascal and $F$-Fermat matrices correspondingly as indicated earlier though in our friendly-mnemonic notation they look so much alike . Namely , the corresponding $K_{\hat{q}_\psi}$ matrix with the Fibonomial choice $\psi_n(q) = [F_n!]^{-1}$ is now of the form

$$
K_{\hat{q}_F} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 7_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11_{\hat{q}_F} & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (n-1)_{\hat{q}_F} & 0
\end{bmatrix}
$$

**Fig.5. The $K_{\hat{q}_F}$ matrix**

Similarly to the earlier case considered $K^n_{\hat{q}_F} = 0; K^k_{\hat{q}_F} \neq 0$ for $0 \leq k \leq (n-1)$ and again we also have

$$
P_{\hat{q}_F}[x] = exp_\psi\{xK_{\hat{q}_F}\} = \sum_{k \in Z_n} \frac{x^k K^k_{\hat{q}_F}}{k_F!}.
$$

The result $P_{\hat{q}_F}[x]$ of the $F$-exponentiation above has been shown on the Fig.6.

$$
P_{\hat{q}_F}[x] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_\psi & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^2 & 2_\psi x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^3 & 3_\psi x^2 & 3_\psi x & 1 & 0 & 0 & 0 & 0 & 0 \\
x^4 & 4_\psi x^3 & 6_\psi x^2 & 4_\psi x & 1 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & 0 \\
x^{n-1} & 0 & 0 & 0 & 0 & 0 & 0 & (n-1)_\psi x & 1
\end{bmatrix}
$$

**Fig.6. The $P_{\hat{q}_F}[x]$ matrix**

**Important-Conclusive.** Apart then from $\psi$-Pascal one source matrix factory of identities we indicate in explicit also the origins of the $\hat{q}_F$ - Pascal and $\hat{q}_F$-Fermat matrices factory of mnemonic attainable identities (compare via [16] with [9-14,18-21,4] ). From operator identities involving the $\hat{q}_\psi$-Pascal $P_{K_{\hat{q}_\psi}}[x]$ and $K_{\hat{q}_\psi}$-Fermat matrix we obtain identities in terms of objects on which the $\hat{q}_\psi$ (or $\hat{q}_{\psi,Q}$ from [3,5,6,16]) act and these are polynomials from F[x] or in more general setting [6,5,3] from formal series algebra F[[x]] where F denotes any field of zero characteristics. In order to get such countless realizations of operator identities in terms of formal series it is enough to act by both sides of a given operator identity on the same element from $F[[x]]$.

# 4 IV. Remark on perspectives

The perspective of numerous applications are opened. Apart from being the natural one source factory of identities $\psi$-Pascal $P_\psi[x]$ and $\hat{q}_\psi$-Pascal $P_{K_{\hat{q}_\psi}}[x]$ and $K_{\hat{q}_\psi}$-Fermat

matrices as well appear to be the similar way natural objects and tools as the Pascal matrix P[x] is in the already mentioned and other applications - (see [4,18]- for example). Just to indicate few more of them: the considerations and results of [4] concerned with Bernoulli polynomials might be extended to the case of $\psi$-basic Bernoulli-Ward polynomials introduced in [1] and investigated recently in [17] in the framework of the $\psi$- Finite Operator Calculus [2,3,5-7] due to the use of the $\psi$- integration proposed in [2,6] . The same applies equally well to the case of $\psi$-basic Hermite-Ward polynomials and other examples of $\psi$-basic generalized Appell polynomials [3,2,5-6] which - being of course $\psi$- Sheffer are characterized equivalently by the familiar $\psi$-Sheffer identity [3,2]

$$A_n(x +_\psi y) = \sum_{k\geq 0} \binom{n}{k}_\psi A_k(y) x^{n-k}. \tag{11}$$

For further possibilities - see references [8-14,18-21] and many other ones not known for the moment to the present author.

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